

Collaborative Problem-Solving Phase in Mathematics: A Case Study in Geometric Problem-Solving

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Abstract: Problem-solving is an activity that can support knowledge construction. Polya, Mason, and Schoenfeld see the problem-solving process as an activity carried out in an individual context. A person's activities cannot be separated from individual interactions. Collaborative Problem-Solving (CPS) is a situation where students must use problem-solving and communication skills in group work to solve a problem. This research examines the stages of collaborative problem-solving in case studies of geometric problem-solving. This research is a case study type with a qualitative approach. The research subjects were class X high school students. The research instruments used were collaborative plane geometry problem-solving questions and interview guidelines. The data collection method was carried out by providing collaborative problem-solving tasks and interviews. The recording was carried out during the assignment using an audio recorder to see the collaborative mathematical problem-solving phase. The research results show that problem-solving starts with discussing, conjecturing, building solutions, communicating reasoning, and extending. The problem-solving phase is cyclical at certain stages, namely discuss – conjecture – build solution – discuss. Meanwhile, critique merges the discussion, conjecture, solution-building, and communication reasoning stages. The finding of the CPS phase in this research is that it has cyclical stages. It combines the critique stages in each CPS phase and can be considered if different CPS phases are documented. Apart from that, it can be a consideration for students that the problem-solving stages in collaboration are more flexible.

Keywords: Collaborative problem solving; phase; geometric; mathematics.

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INTRODUCTION

Knowledge construction activities are the right way to learn mathematics. (Acharya et al., 2022; Guerrero, 2021). Problem-solving is an activity that can support knowledge construction (Charlesworth & Leali, 2012). Problem-solving is using previously learned knowledge, understanding and skills to meet the demands of "foreign" conditions (Brehmer, 2015) for example, facing problems that do not have clear procedures or algorithms so that solutions cannot be identified directly (Siniguan, 2017). In this research, problem-solving is associated with mathematics education. Problem-solving occurs when students are given a problem whose solution does not have a specific algorithm or is non-procedural (Byrne, 2013). Singh et al., (2018) pioneered the development of problem-solving heuristic stages in mathematics, which was influential and is still relevant in modern mathematics. Thiangtung (2016,) expanded Polya's problem-solving stages by responding to being "stuck" in the problem-solving process. Furthermore, Fensel et al., (2002) also introduced a normative problem-solving structure. However, all three see the problem-

solving process as an activity carried out in an individual context. A person's activities cannot be separated from individual interactions. Collaborative Problem-Solving (CPS) is a situation where students must use their problem-solving and communication skills in group work to solve a problem (Häkkinen et al., 2017).

Collaborative problem-solving skills are essential for students in the twenty-first century (Fitzsimons & Ní Fhloinn, 2023; Olivares et al., 2021). The interactions in CPS positively impact the cognitive, metacognitive and affective domains because they trigger students to think critically and logically (Gokhale, 1995; Kyndt et al., 2013; Xu et al., 2023). In CPS, team members build a shared understanding to solve problems (Roschelle & Teasley, 1995). All problem-solving activities, such as setting goals and interpreting problems, plans, and problem-solving actions, are negotiated in a social context (Roschelle & Teasley, 1995). Roschelle (1992) states that success in building shared understanding is the existence of repeated cycles of confirming and improving shared understanding. In other words, the stages are given a problem.

Collaborative problem-solving is a cyclic stage, which refers to the stages of individual problem-solving. However, Roschele's opinion is contrary to Artzt & Armour (1992), which states that the stages of collaborative problem-solving do not occur cyclically and are linear, which refers to the individual solving framework proposed by Polya (Polya, 1945) and Schoenfeld (Schoenfeld, 1985). The stages of collaborative problem-solving are not cyclical because there is space for interaction between team members. The results of research conducted by Salminen-Saari (2021) state that the stages of collaborative problem-solving are non-cyclical and recursive. Considering these differences in views and the lack of empirical evidence, this research aims to examine the stages of collaborative problem-solving in case studies of geometric problem-solving. Considering that working collaboratively is an activity that is often implemented in the classroom as a result of the demands of collaboration skills in the 21st century, the description of the stages of collaborative problem-solving needs to be clarified to guide teachers and researchers in understanding the process of collaborative problem-solving.

Problem-solving activities are complex because they require the use of intellectual abilities such as perception, memory, reasoning, and self-control (Căprioară, 2015). In mathematics, geometry is one of the basic materials for studying spatial visualization. The basic concept of geometry is closely related to shape, size and distance (Walle et al., 2016). Solving geometry-related problems requires geometric reasoning and spatial estimation skills (Shi et al., 2023) so problem-solving activities will be seen more clearly. Apart from that, solving geometric problems has been proven to trigger collaboration between team members based on research conducted by (Bjuland, 2004; Oner, 2013; Pöysä-Tarhonen et al., 2022). Therefore, it is necessary to study further the collaborative solution stages carried out by students in solving geometric problems.

Theoretical Framework

Problem-solving is the main focus of mathematics education (Carrillo & Cruz, 2016; Schoenfeld, 2013). Schoenfeld (1983) defines problem-solving as finding solutions to problems that do not have a specific solution algorithm. Problem-solving is essential to mathematics (Halmos, 1980) and mathematics learning (Liljedahl et al., 2016). Apart from that, problem-solving helps help students learn to think mathematically (Schoenfeld, 1985).

Polya (1945) was the first pioneer of the stages of mathematical problem solving, which consisted of understanding the problem, preparing a plan, implementing the plan, and looking back. Solving mathematical problems relies on the first stage: understanding the problem. Apart from being able to interpret problems, students at this stage are also required to have the desire to solve problems (Bell & Polya, 1945). This means a problem must be structured well to make problem-solving activities practical (Jäder et al., 2017). Polya's problem-solving stages have become a basic concept for current researchers. Schoendfeld (2016) states that mathematical problem-solving is a cognitive process of using knowledge and developing an understanding of a topic. In other words, this cognitive process is not just a procedural repetition exercise.

Hesse et al (2015) define collaborative problem-solving as a problem-solving activity by exchanging ideas. OECD (2015) defines collaborative problem-solving as the capacity of individuals to engage in joint problem-solving by sharing understanding, knowledge and skills to solve a given problem. For collaborative problem-solving to be successful, Rosen (2014) states that teams must develop a shared understanding of the problem and maintain it throughout the problem-solving process. Additionally, the success of collaborative problem-solving depends on the communication skills of each team member (Hesse et al., 2015).

Similar to individual problem-solving, collaborative problem-solving also has several phases. Several researchers have identified various phases of collaborative problem solving, including Artz & Armour (Artzt & Armour-Thomas, 1992), whose stages are hierarchical and not cyclical; Salminen-Saari (Salminen-Saari et al., 2021) is recursive and not cyclical; and Fitzsimons & NiFhloinn (Fitzsimons & Ní Fhloinn, 2023) are hierarchical and cyclical. Figure 1 represents each phase of collaborative problem-solving.

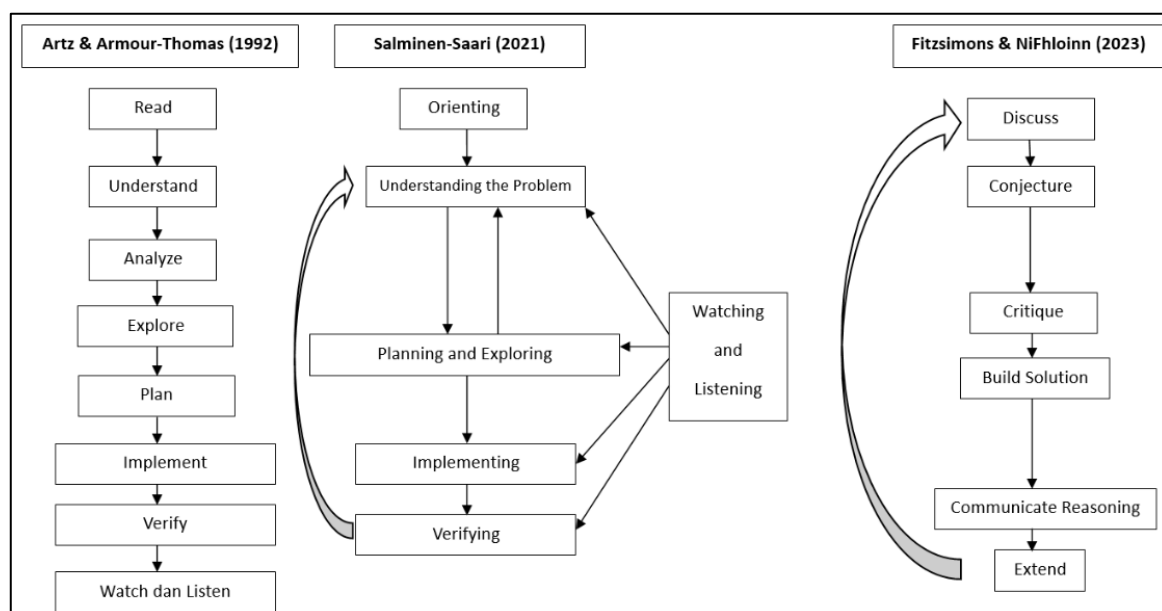


Figure 1. Collaborative Problem-Solving Phase

METHOD

This research describes the collaborative mathematical problem-solving phase in geometric problems. This research is a qualitative case study type of research. The research subjects were a group of students consisting of two class X high school students who had

studied flat geometry material. Zuniga et al (2021) suggested that collaborating in pairs can enhance group members' engagement in negotiation, interaction, consensus-building, and evaluation. As a result, CPS is more likely to function effectively. The participants were selected according to their mathematical proficiency, which was assessed based on their final exam scores in flat geometry. A selected group consisted of two students with high mathematics abilities. Next, the two students are called T1 and T2. We chose the combination of group members because it is an ideal group for representing the CPS process. Besides that, subject selection by considering the closeness between students. The closeness between students is considered so that communication between students runs smoothly and the collaboration intensity will be high. Hakkinen et al (Häkkinen et al., 2017) stated that communication between groups is the key to successful collaborative problem-solving. The subjects selected in this research were groups with a high collaboration intensity. The group selected as the subject consists of members who can work together and are committed to completing the assigned task. The ability to collaborate is defined as the fulfilment of three activities in collaborative problem-solving, namely communication, negotiation, and agreement, during the problem-solving process. Meanwhile, sincerity is reflected in three main aspects: persistence, focus, and the ability to produce meaningful answers. Perseverance means not giving up easily when facing difficulties in solving a problem. Focus indicates that the individual remains concentrated without being distracted by other factors during the solving process. Meanwhile, meaningful answers indicate the results of mature thinking, not just completing or filling in answers without consideration. This is done so that each problem-solving phase is carried out in collaboration with group members, obtaining rich data about the collaborative problem-solving phase.

The research instruments used were collaborative flat-plane geometry problem-solving tasks and interview guidelines. The data collection method was done by giving assignments, observations and interviews. Observation collects data through conversations and things students do while completing tasks collaboratively. To ensure that no group activities or conversations are missed to be observed, all student activities are recorded using audio-video recorder media during the problem-solving process. Researchers employed a task-based semi-structured interview method, where questions were designed based on the tasks performed. This approach was selected to investigate collaborative mathematical problem-solving phase, which are not explicitly visible in CPS transcripts. For instance, researchers needed to determine whether new ideas stemmed from analyzing provided information or were merely based on assumptions. The following is a collaborative problem-solving task given.

A tubular has a height of 9 cm and a base diameter of 12 cm. A ball with a diameter of 6 cm is placed in a can. How many balls can be put in the can? Give reasons.

Data analysis was carried out after transcription of the audio recorder recording results. Data analysis was carried out in four stages, namely: 1) Category Construction is the coding of transcription results; 2) Sorting Categories and Data is selecting data unrelated to the collaborative problem-solving phase; 3) Naming the Categories is categorizing the selected data; 4) Theorizing is connecting several categories of data to obtain data for the collaborative problem-solving phase. The data analysis process was carried out on group

work data when completing assignments, discussion recordings, and transcripts of group interviews. Initial data condensation was done by coding each group's collaborative problem-solving stages. The collaborative problem-solving stage codes were arranged based on the CPS stage indicators in Table 1. The coding of the CPS stages was carried out to understand the collaborative problem-solving process that occurred in each group. Data analysis uses the collaborative problem-solving phase proposed by Fitzsimons & NiFhloinn (Fitzsimons & Ní Fhloinn, 2023) because apart from joint activities, this phase also considers the activities of each team member. According to Clarke (2011), individual-based problem-solving is the basis for forming collaborative problem-solving. Table 1 is a description of the phases proposed by Fitzsimons & NiFhloinn (Fitzsimons & Ní Fhloinn, 2023).

Tabel 1. Collaborative Problem-Solving Phase Description

Fase	Description
Discuss (D)	- Read the problem individually (D1) - Discuss about: (D2) - Interpretation of the problem - Confirming various misconceptions that arise - Possible approach to use
Conjecture (Co)	- Choose to prove or disprove the ideas that arise (Co1) - Team members carry out problem-solving experiments both individually and in groups. (Co2)
Critique (Cr)	- Self-regulation of the results of self-solving experiments (Cr1) - Criticize the work of other team members (Cr2)
Build Solution (BS)	- Gather all information relevant to the solution (BS1) - Check the solution (BS2)
Communicate Reasoning (CoR)	- Discuss the problem-solving process again (CoR1) - Explain the solution to group members (CoR2)
Extend (E)	- Re-examine the critical points of the solution found (E1) - Assess errors during the problem-resolution process (E2)

RESULT AND DISCUSSION

The results of observations of CPS activities in a case study of solving geometric problems are presented below. After the group received the assignment, both T1 and T2 looked at the assignment given. This marks that the subject has entered the *discussion stage*. The following is the conversation between T1 and T2 at the discussion stage.

T1 : (reading the question) This is calculated using the usual formula, right? [D1]

T2 : Yes [D1]

T1 : (calculating by drawing three balls in a triangular arrangement) [D2]

T2 : (calculating the volume of the ball and the volume of the cylinder) [D2]

T1 : The base diameter is 12, and the ball diameter is 6. For example, there are already two balls inserted, then the other ball is placed in the middle like this. **This means that the ball is in a slightly lower position. So, its height will decrease.** [D2]

T2 : Yes, that's right, its height will decrease.

- T1 : But isn't it wrong to solve it that way? **Or using a triangle like this? like the one usually found in other questions?** (while drawing a triangle surrounding the three balls drawn at the beginning). [D2]
- T2 : Yes, you can also use a triangle like that.

In the discussion stage, T1 interprets the problem by drawing a tube filled with two balls. Next, T1 measures the height of the remaining tubes and adds one ball to the picture. The visualization carried out by T1 is based on known data: the tube has a base diameter of 12 cm, and the ball has a diameter of 6 cm. At the same time, T2 calculates the cylinder's volume and the ball's volume. T1 then explained the results of his interpretation to T2. T1 stated that T1 had difficulty calculating the number of balls that could be put into the tube. In response, T2 tried to draw three balls arranged in a triangle shape to interpret the problem presented by T1 as in Figure 2. T2 did this based on the design of a mathematics problem he had worked on previously.



Figure 2. Visualization of T2 in understanding the interpretation of the T1 problem

T1 agrees with the image produced by T2. T1 and T2 agree that the number of balls can be determined by determining the value (height of the third ball as in Figure 2). Next, the group tries to determine the value. This marks that the group has entered the conjecture stage. The following is the conversation between T1 and T2 at the conjecture stage.

- T1 : But this can... (thinking). **This can use Pythagoras, right?** The length of this side is 3,3,3, so the length of this side is $3\sqrt{3}$. [Co2]
- T2 : So that means the length of this part is $6-3\sqrt{3}$. That means **the first step is two balls, then one more ball can be placed to the right and left, not right in the middle like this, but forward or backwards.** [Co2]
- T1 : Forward or backward? (thinking)
- T2 : Yes, that's right. But it can't be because this part is already 6 long. The two balls here can't move anymore. That means we now calculate this (pointing to the area of the third ball that fits between the two balls). That means $3\sqrt{3}$ minus... (cut by T2)
- T1 : 6 minus $3\sqrt{3}$
- T2 : What's the point of 6? This part has a length of $3\sqrt{3}$.
- T1 : The cylinder has a height of 9, while this part has a length of 6 (while pointing to the height of the ball).
- T2 : **Doesn't this part need to be calculated?** (pointing to the height of the third ball that doesn't fit between the two balls). [Co1]
- T1 : It's calculated.
- T2 : That means 9 minus 6, so there are 3 left.
- T1 : **Why 9? Why not 6 minus $3\sqrt{3}$?** [Cr2]

- T2 : This part is 9 high, and this part is 3. That means 9 minus 6 equals 3. That means it's not enough for three balls. **That means it's only enough for two balls. [Co2]**
- T1 : enough for three
- T2 : why can it be 3?
- T1 : the circle has a diameter of 6, then minus $3\sqrt{3}$ because this part has a length of $3\sqrt{3}$.
- T2 : **But this part is not as long as $3\sqrt{3}$. [Cr2]**
- T1 : Yes, that's right.

T1 tries to determine the value using the Pythagorean formula. First, T1 connects the radii of the two balls to determine the height of PQ, as in Figure 3 below.

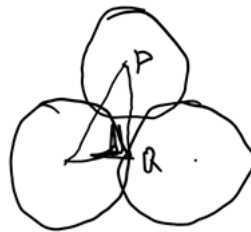


Figure 3. Visualization of T1 at the conjecture stage

From the results of the T1 calculation, the length of PQ is $3\sqrt{3}$. Next, T2 states that the length h is $(6 - 3\sqrt{3})$. But T1 criticized T2's answer. T1 states that $3\sqrt{3}$ is the length of PQ, which differs from the length of the diameter minus the length h . T2 agreed with the rejection of T1's idea. Furthermore, T2 proposed that to determine the number of balls that can fit in the tube, reduce the height of the tube by the diameter of the ball, namely $9 - 6 = 3$. So, the remaining height of the tube is 3 cm, whereas if the third ball is inserted, the tube cannot be closed entirely. In other words, only two balls can be put into the tube. Next, to confirm the answer, T1 tried to use another solution approach proposed by T2 at the discussion stage, namely using volume. Thus, the subject enters the *build solution stage*, namely collecting all relevant information. The following is the conversation between T1 and T2 at the build solution stage.

- T1 : **What do you think? Is it necessary to use the volume formula? [BS2]**
- T2 : You can't use the volume formula because if a ball is put in a tube, a cavity will be left.
- T1 : Right. **But how about trying it first?** (*calculating the volume of the tube and the volume of the ball*). If you use the volume formula, the result is 9. [BS1]
- T2 : **These two balls can fit in the tube, but for the third ball...** (*thinking*) [Cr2]
- T1 : Like this, **first, we show the solution using the volume formula. Then we add the following explanation:** "You can't use the volume formula because placing the ball in the tube will, of course, create empty space". [BS1]
- T2 : **This question is indeed a trap for using volume. [Cr1]**
- T1 : So, what happens? Do we use the volume formula or the results of our previous calculations?
- T2 : But if we use our previous calculations, we don't find the exact calculation.

At the build solution stage, T1 divides the volume of the tube by the volume of the ball, producing a ball that can fit into the tube. T2 criticized the answer from T1 because it was illogical based on the fact that if a ball is put into a tube, there must be space. So, you cannot use volume calculations. T2's rejection of the alternative solution presented by T1 triggers T1 to re-examine the problem-solving process by interpreting the problem. Thus, the subject enters the *discussion stage* again. The following is the conversation between T1 and T2 at the second discussion stage.

- T1 : **Take a look at this (drawing). The bottom of the tube has a diameter of 12, while the diameter of the ball is 6. That means four balls can fit in. [D2]**
- T2 : **The 4 balls are at the same height level, right? [D2]**
- T1 : Yes. What do you think? Like this (*drawing four balls*). This is a perfect circle if divided into 4.
- T2 : 4 balls can fit in perfectly; what do you mean?
- T1 : Yes, 4 balls can fit in perfectly. **Can't these four balls be stacked again? [Cr1]**
- T2 : **Yes. There is a space in the middle here so the fifth ball can fit in even more profoundly. [Cr2]**
- T1 : Yes, it can. Then why couldn't it fit two balls?

In this second discussion stage, T1 interpreted the problem by drawing the base of a circular tube with a diameter 12 and placing four circles with a diameter of 6. Thus, the results of T1's interpretation found that four balls could be put into the tube. Next, T2 redraws the interpretation of T1. T2 rejected T1's answer because, based on the visualization that T2 had created, the base of the tube was a circle. If the base of the tube is square, it is still possible to place four balls. T2's rejection was accepted by T1. Next, T2 re-examines several solutions that have been found for comparison. Thus, the group entered the *communication reasoning stage*. The following is the conversation between T1 and T2 at the communication reasoning stage.

- T2 : 4 balls can be placed on the same level.
- T1 : I'm sure it can because it fits if we draw it precisely according to the size.
- T2 : Right (**checking again by drawing according to the precise size**) [CoR1]
You're wrong (*showing the drawing to T1*)
- T1 : **Yes, it can't. The base is not square but a circle so that the tip of the ball can come out of the tube.** Then how? [CoR1]
- T2 : **See, is the slope of this ball the same as that of a quarter circle?** [CoR2]
- T1 : The slope of the ball doesn't seem to matter because the slope is inside the ball. What matters is the middle part towards the tip.
- T2 : **The length of this part is 6, this part is 6, and this part is 12.** [CoR2]
- T1 : Yes, the length of the base diameter is 12
- T2 : **That means this part has a length of 6, and so does this part, meaning that these 4 balls can't be put in.** [CoR2]
- T1 : That means two balls, right?
- T2 : **3 balls can still fit in. There is a space here that can be filled with one more ball (pointing to the empty area)** [Cr2]

- T1 : Then we must **measure this part to ensure one ball can be put in**. Can this section be 5? [Cr2]
 T2 : **How long will this be?** [Cr2]
 T1 : **I can't be sure yet. However, two balls will fit in. But 3 balls? 4 balls? 5 balls? I can't be sure.** [Cr1]

The communication reasoning stage begins when T1 compares the problem-solving process that has been carried out. In solving the first problem, it was found that three balls could not be put into the tube. This happens because the remaining height of the tube after being filled with two balls is 3 cm. Meanwhile, the diameter of the third ball is 6 cm. Thus, according to T1, the tube can only be filled with two balls. In the process of solving the second problem, a tube and ball volume approach was used, and it was found that dividing the volume resulted in 9 balls that could be put into the tube. But this solution does not match the facts because a ball inserted into the tube will leave some space in the tube.

Meanwhile, while solving the third problem, it was found that four balls could not be put into the tube. This happened because of a visualization error made by T1. Based on a comparison of several problem-solving processes, T1 stated that only two balls could be put into the tube. T1's statement was not approved by T2 because, according to T2, there was still a chance for a third ball to be inserted. After all, there was a space between the two balls, which could be partially filled by a third ball so that the third ball could be put into the tube. But T2 cannot prove with certainty the height of the ball that can fill the space between the first and second ball. Responding to T2's statement, T1 then re-examined several solutions that had been found. So, the group enters *the extended stage*. The following is the conversation between T1 and T2 at the extended stage.

- T1 : What if the answer is "less than 9 balls can be put into the tube"? **This answer can be combined with the volume formula and our earlier calculations.** [E1]
 T2 : **Ehm, answer that two balls can fit in because the diameter is 6.** [E1]
 T1 : However, **1:2 is that so?** [E2]
 T2 : **1:2 what? The comparison between what and why must be clear.** [E2]
 T1 : Then what?
 T2 : I agree with your answer earlier, namely that less than 9 balls can be put into the tube.

In the extended stage, both T1 and T2 record the solutions found. The solutions found are 1) based on the size of the tube and ball, two balls can be put into the tube, and 2) based on volume measurements, nine balls can be put into the tube. Thus, T1 and T2 agreed that the answer to the task was that the number of balls that could be put into the tube was less than nine. This answer was chosen because it has a strong reason, namely that there is still space in the tube filled with balls.

Meanwhile, the first answer still has another possible answer, namely three balls, but it cannot be proven accurately. Table 2 is a summary of collaborative problem-solving process activities.

Table 4. Collaborative Problem-Solving Process

Sequence of Activities	Activities	CPS Phase	The results of CPS activities on the progress of problem-solving
1	Provide collaborative problem-solving assignments.	-	
2	a. T1 represents the problem by visualizing the problem-solving task. b. T2 represents the problem by calculating the respective volumes of the cylinder and sphere. T2 represents the interpretation results of T1 again.	Discuss	T1 and T2 agree that the number of balls can be determined by first determining the height of the third ball, which does not fit in the space between the first and second balls (h).
3	a. T1 determines the value of h using Pythagoras. T1 finds the length of PQ is $3\sqrt{3}$. T2 says the length of h is $(6 - 3\sqrt{3})$. b. T1 criticizes T2's answer (part b) because $3\sqrt{3}$ is the length of PQ, which differs from the diameter of the third sphere minus the length of h .	Conjecture Critique	Length of h adalah $(6 - 3\sqrt{3})$. T1 and T2 do not find the length of h .
4	T2 proposes subtracting the height of the tube from the diameter of the ball to determine the number of balls that can be inserted into the tube.	Conjecture	The answer is that two balls can be put into the tube (based on T2's suggestion).
5	a. T2 tries to check the answer by dividing the cylinder's volume by the ball's volume. b. T1 criticized T2's answer because it was based on the fact that if a ball is put into a tube, there must be space.	Build Solution Critique	The answer is that nine balls can be put into the tube. T1 and T2 have yet to find a definite answer for how many balls can be put into the tube.
6	a. T1 interprets the problem again by drawing the base of a circular cylinder with a diameter 12 and placing four circles with a diameter 6. b. T2 re-visualized the interpretation of T1's problem, and T2 rejected T1's answer because, based on the visualization, the base of the tube was a circle, so four balls could not be put in the tube.	Discuss Critique	The answer is that four balls can be put into the tube. T1 and T2 have not found a definite answer for how many balls can be put into the tube.
7	a. T1 compares the problem-solving process carried out. b. T2 disagrees with T1's conclusion. T2 believes that three balls can be put into the tube because there is a space between the two balls, which can be filled by a third ball so that the third ball can be put into the tube.	Communicate Reasoning Critique	The answer is that two balls can be put into the tube. T1 and T2 agree that there is a chance that three balls can be put into the tube, but T1 and T2 cannot prove it.
8	T1 and T2 list several possible solutions found.	Extend	T1 and T2 agreed that the answer to the given task was that the number of balls that could be put into the tube was less than nine.

Based on Table 2, the chart in Figure 4 represents the collaborative problem-solving phase in the geometric problem-solving case study.

The results of case study research on solving geometric problems show cyclical stages in the CPS problem-solving process. The cyclical stages that occur are *discuss-conjecture-build solution-discuss*. Discuss is a stage related to the activity of interpreting a given problem. Interpreting the problem will be carried out again by the problem solver when they experience problems in determining the solution to the problem. Barham (2020) states that correct problem interpretation is one of the keys to finding a solution to a problem. Thus, when a solution has not been found, the problem solver will reinterpret the problem. In the context of collaborative work, the team will reinterpret the problem or task given to ensure accuracy in understanding the problem. Gok & Sylay (2010) stated that to choose the right problem-solving method, students must be aware of what they are doing and why. Therefore, when a solution is not found, students will repeatedly carry out the problem interpretation stage. Roschelle (Roschelle & Teasley, 1995) also states that success in building shared understanding is the existence of repeated cycles of confirming and improving shared understanding.

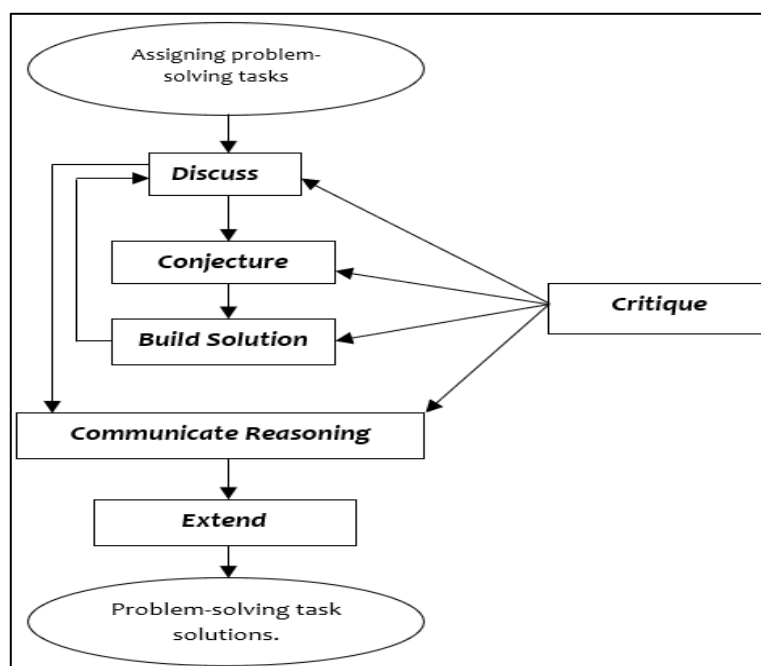


Figure 4. Collaborative Problem-Solving Phase Chart in Solving Geometry Problems.

Another finding in this research is the critique stage, which always appears at the discuss, conjecture, build solution and communicate reasoning stages. Based on Polya (Polya, 1945) stages of problem-solving, the four stages can be categorized as understanding the problem, planning, implementing the plan, and conducting the evaluation. In a collaborative context, responding to each other regarding approval or rejection cannot be avoided at every problem-solving stage. Chan & Clarke (2017) stated that interactions responding to ideas, suggestions and findings related to problem-solving occur in collaborative dialogue between students. The responses given to each other are in the context of self-regulation of teamwork and independent work results. Therefore, *critique* will appear at every stage of problem-solving. Based on these findings, the

researcher found that *critique* is not part of the CPS stages but is a result of the collaborative context. In other words, *critique* will emerge automatically in a collaborative context due to interactions between team members.

This research is limited to only one group, with the selection of group member partners based only on personal closeness between group members. It cannot be denied that students' knowledge of the material influences the success of problem-solving. Yew & Zamri (2016) stated that a lack of understanding of the material results in problem-solving using an unsystematic trial-and-error strategy, so it will be challenging to find solutions. Therefore, mathematical ability, apart from paying attention to interpersonal closeness, is also considered in future research. Even though it has limitations, the findings of this research can be used as a reference for teachers or researchers who choose CPS as a learning method. The finding of the CPS phase in this research is that it has cyclical stages. It combines the *critique stages* in each CPS phase and can be considered if different CPS phases are documented. Apart from that, it can be a consideration for students that the problem-solving stages in collaboration are more flexible.

CONCLUSION

The collaborative problem-solving phase consists of six stages: *discuss, conjecture, critique, build solution, communicate reasoning, and extend*. The six stages are cyclical. The empirical results of this research reveal that the *critique stage* is found at each *discussion, conjecture, build solution, and communication reasoning stage*. In collaboration, *critique* is responding, reacting and self-regulating the problem-solving process by oneself or other team members. In this way, *critique* is not a stage but an activity that occurs in collaborative problem-solving. Apart from that, the collaborative problem-solving phase was found to be cyclical at certain stages, namely *discuss-conjecture-build solution-discuss*. The finding of the CPS phase in this research is that it has cyclical stages. It combines the *critique stages* in each CPS phase and can be considered if different CPS phases are documented. Apart from that, it can be a consideration for students that the problem-solving stages in collaboration are more flexible. This study is limited to only one group, and the selection of group members is based on mathematical ability and personal closeness between group members. Undeniably, many influential variables in collaborative problem-solving will emphasize social interaction. For example, in the cognitive domain, there is a composition of mathematical ability between members; in the affective domain, self-efficacy and critical thinking disposition can affect the collaborative problem-solving process. This can be a consideration for further research.

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